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Frames of multipliers in tensor products of Hilbert modules over pro-C^*-algebras

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ABSTRACT

Given a standard frame of multipliers $\{h_n\}_n$ for a Hilbert A-module E and a standard frame of multipliers $\{t_n\}_n$ for a Hilbert B-module F we construct a standard frame of multipliers for the external tensor product of E and F and for the inner tensor product of E and F using a non-degenerate pro-C^*-morphism φ from A to the pro-C^*-algebra of all adjointable module morphisms on F.

1. Introduction

The notion of frames in Hilbert spaces was introduced by R.J. Duffin and A.C. Schaeffer [2] in the context of non-harmonic Fourier series. M. Frank and D. Larson [4,5] generalized this notion to the situation of Hilbert C^*-modules. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. But the theory of Hilbert C^*-modules is different from the theory of Hilbert spaces, for example, no any closed submodule of a Hilbert C^*-module is complemented.

M. Frank and D. Larson [4,5] showed that any countably generated Hilbert C^*-module possesses a standard frame, but the Hilbert C^*-modules A and H_A are countably generated if and only if the C^*-algebra A has a countable approximate unit. I. Raeburn and S.J. Thompson [13] considered a more general notion of countably generated Hilbert C^*-modules in which generators are multipliers of the module. Also they introduced the notion of standard frame of multipliers for a Hilbert C^*-module and proved some basic properties of standard frames of multipliers. In this paper we consider the standard frames of multipliers in Hilbert modules over pro-C^*-algebras [8] and investigate the standard frames of multipliers in tensor products of Hilbert modules.

The paper is organized as follows. In Section 2 we recall some facts about pro-C^*-algebras and Hilbert modules over pro-C^*-algebras. In Section 3, we show that an isomorphism of Hilbert modules preserves standard frames of multipliers (Theorem 3.4). Also we show that a surjective adjointable module morphism preserves standard frames of multipliers (Proposition 3.3). A. Khosravi, B. Khosravi [10] showed that the tensor product of frames for the Hilbert C^*-modules E and F is a standard frame for the tensor product $E \otimes F$ and the tensor product of the frame operators is the frame operator for the tensor product of frames. In Section 4 we extend the result of A. Khosravi, B. Khosravi for standard frames of multipliers. We show that the tensor product of standard frames of multipliers for E and F is a standard frame of multipliers for $E \otimes F$ and the tensor product of the frame operators is the frame operator for the tensor product of standard frames of multipliers (Theorem 4.2). Section 5 is devoted to standard frames of multipliers in inner tensor product of Hilbert modules over pro-

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C^*-algebras. Given two standard frames of multipliers for the Hilbert modules E and F which verify some conditions we construct a standard frame of multipliers in the inner tensor product of E and F (Theorem 5.3). Also, we obtain a standard frame of multipliers for E and F from a standard frame of multipliers for the inner tensor product of E and F under some conditions (Propositions 5.5 and 5.6).

2. Preliminaries

Pro-C^*-algebras are generalizations of C^*-algebras. Instead of being given by a single C^*-norm, the topology on a pro-C^*-algebra is defined by a directed family of C^*-seminorms. Clearly, any C^*-algebra is a pro-C^*-algebra. The class of pro-C^*-algebras is bigger than the class of C^*-algebras, for example, the $*$-algebra $C_{cc}([0,1])$ of all complex valued continuous functions on $[0,1]$ with the topology of uniform convergence on the countable compact subsets of $[0,1]$ is a pro-C^*-algebra which is not topologically isomorphic with any C^*-algebra (see, for example, [3]). In the literature, pro-C^*-algebras have been given by different names such as b^*-algebras (C. Apostol), LMC^*-algebras (G. Lessner, K. Schmüdgen) or locally C^*-algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc.).

Let A be a pro-C^*-algebra. The set $S(A)$ of all continuous C^*-seminorms on A is directed with the partial order given by $p \geqslant q$ if $p(a) \geqslant q(a)$ for all $a \in A$ and it determines the topology on A in the sense that a net $\{a_i\}_{i\in I}$ converges to 0 in A if and only if the net $\{p(a_i)\}_{i\in I}$ converges to 0 for all $p \in S(A)$.

An element $a \in A$ is *bounded* if $\|a\|_\infty = \sup\{p(a);\ p \in S(A)\} < \infty$. More information about pro-C^*-algebras in [3,12].

A *pre-Hilbert A-module* is a complex vector space E which is also a right A-module, compatible with the complex algebra structure, equipped with an A-valued inner product $\langle \cdot,\cdot \rangle : E \times E \to A$ which is $\mathbb{C}$- and A-linear in its second variable and satisfies the following relations:

(1) $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ for every $\xi, \eta \in E$;
(2) $\langle \xi, \xi \rangle \geqslant 0$ for every $\xi \in E$;
(3) $\langle \xi, \xi \rangle = 0$ if and only if $\xi = 0$.

We say that E is a *Hilbert A-module* if E is complete with respect to the topology determined by the family of seminorms $\{\overline{p}_E\}_{p\in S(A)}$, where $\overline{p}_E(\xi) = \sqrt{p(\langle \xi, \xi \rangle)}$, $\xi \in E$.

An element ξ in a Hilbert A-module E is *bounded* if $\|\xi\|_\infty = \sup\{\overline{p}_E(\xi);\ p \in S(A)\} < \infty$.

If A is a pro-C^*-algebra, then A is a Hilbert A-module with $\langle a, b \rangle = a^*b$, and the set H_A of all sequences $(a_n)_n$ with $a_n \in A$ such that $\sum_n a_n^* a_n$ converges in A is a Hilbert A-module with the action of A on H_A and the inner product defined by $(a_n)_n b = (a_n b)_n$, respectively $\langle (a_n)_n, (b_n)_n \rangle = \sum_n a_n^* b_n$.

Let E and F be two Hilbert A-modules. A map $\Phi : E \to F$ is *a morphism of Hilbert modules over pro-C^*-algebras* if there is a morphism of pro-C^*-algebras $\varphi : A \to B$ such that

$$\bigl\langle \Phi(\xi), \Phi(\eta) \bigr\rangle = \varphi\bigl(\langle \xi, \eta \rangle\bigr) \quad \text{for all } \xi, \eta \in E.$$

We say that Φ is *an isomorphism of Hilbert modules over pro-C^*-algebras* if it is invertible and Φ^{-1} and Φ are morphisms of Hilbert modules over pro-C^*-algebras.

A module morphism $T : E \to F$ is *continuous* if for each $p \in S(A)$ there is $M_p > 0$ such that $\overline{p}_F(T\xi) \leqslant M_p \overline{p}_E(\xi)$ for all $\xi \in E$, and it is *adjointable* if there is a module morphism $T^* : F \to E$ such that $\langle T\xi, \eta \rangle = \langle \xi, T^*\eta \rangle$ for every $\xi \in E$ and $\eta \in F$. Any adjointable module morphism is continuous. The set of all adjointable module morphisms from E to F, denoted by $L_A(E,F)$, is a locally convex space with respect to the topology determined by the family of seminorms $\{\widetilde{p}_{L_A(E,F)}\}_{p\in S(A)}$, where $\widetilde{p}_{L_A(E,F)}(T) = \sup\{\overline{p}_F(T(\xi));\ \overline{p}_E(\xi) \leqslant 1,\ \xi \in E\}$. Moreover, $L_A(E) = L_A(E,E)$ is a pro-C^*-algebra.

Now we recall some facts about multiplier module from [8,9,13]. *A multiplier* of the Hilbert A-module E is an adjointable module morphism from A to E. The set $M(E)$ of all multipliers of E is a Hilbert $L_A(A)$-module with the action of $L_A(A)$ on $M(E)$ defined by

$$M(E) \times L_A(A) \ni (T, S) \mapsto T \cdot S = TS \in M(E)$$

and the inner-product defined by

$$M(E) \times M(E) \ni (T, R) \mapsto \langle T, S \rangle_{M(E)} = T^*S \in L_A(A).$$

Since the pro-C^*-algebras $L_A(A)$ and $M(A)$, the multiplier algebra of A, can be identified, the Hilbert $L_A(A)$-module $M(E)$ can be regarded as a Hilbert $M(A)$-module and it is called *the multiplier module* of E. Moreover, the topology on $M(E)$ induced by the inner product coincides with the topology defined by the family of seminorms $\{\overline{p}_{M(E)}\}_{p\in S(A)}$, with $\overline{p}_{M(E)}(h) = \widetilde{p}_{L_A(A,E)}(h)$ for all $h \in M(E)$ and for all $p \in S(A)$.

The map $\mathrm{i}_E : E \to M(E)$ defined by

$$\mathrm{i}_E(\xi)(a) = \xi a, \quad \xi \in E,\ a \in A$$

identifies E with a Hilbert submodule of $M(E)$ and then

$$\langle h,\xi\rangle_{M(E)} = h^*(\xi)$$

for all $h \in M(E)$ and $\xi \in E$. Moreover, if $a \in A$ and $h \in M(E)$, then $h \cdot a$ can be identified with $h(a)$.

A sequence $\{h_n\}_n$ in $M(E)$ is *a standard frame of multipliers* for E if for each $\xi \in E$, $\sum_n \langle \xi, h_n\rangle_{M(E)}\ \langle h_n,\xi\rangle_{M(E)}$ converges in A with respect to the topology determined by the family of C^*-seminorms $S(A)$, and there are two positive constants C and D such that

$$C\langle\xi,\xi\rangle \leqslant \sum_n \langle\xi,h_n\rangle_{M(E)}\langle h_n,\xi\rangle_{M(E)} \leqslant D\langle\xi,\xi\rangle$$

for all $\xi \in E$. If $D = C = 1$ we say that $\{h_n\}_n$ is a standard normalized frame of multipliers.

For any positive integer m, the map $e_m : A \to H_A$ defined by

$$e_m(a) = (\delta_{mn}a)_n$$

is a multiplier of the Hilbert A-module H_A and $\{e_m\}_m$ is a standard normalised frame of multipliers for H_A.

3. Frames of multipliers

Proposition 3.1. *Let E be a Hilbert module over a pro-C^*-algebra A and $\{h_n\}_n$ a sequence in $M(E)$. Then $\{h_n\}_n$ is a standard frame of multipliers for E if and only if there is a unique positive invertible element S in $b(L_A(E))$ such that the series $\sum_n h_n \cdot \langle h_n, S(\xi)\rangle_{M(E)}$ converges to ξ in E for each $\xi \in E$. Moreover, $\frac{1}{\|S\|_\infty}$ and $\|S^{-1}\|_\infty$ are the frame bounds.*

Proof. First we suppose that $\{h_n\}_n$ is a standard frame of multipliers for E. Then, by [8, Theorem 4.4], there is a unique positive invertible element S in $b(L_A(E))$ such that the series $\sum_n h_n \cdot \langle Sh_n,\xi\rangle_{M(E)}$ converges to ξ in E for each $\xi \in E$, and since $\langle Sh_n,\xi\rangle_{M(E)} = \langle h_n, S(\xi)\rangle_{M(E)}$, the series $\sum_n h_n \cdot \langle h_n, S(\xi)\rangle_{M(E)}$ converges to ξ in E for each $\xi \in E$.

Conversely, if $\xi \in E$, then

$$\bigl\langle S(\xi),\xi\bigr\rangle = \sum_n \bigl\langle S(\xi),h_n\bigr\rangle_{M(E)} \cdot \bigl\langle h_n, S(\xi)\bigr\rangle_{M(E)}.$$

Therefore, for each $\xi \in E$, the series $\sum_n \langle\xi,h_n\rangle_{M(E)} \cdot \langle h_n,\xi\rangle_{M(E)}$ converges to $\langle \xi, S^{-1}(\xi)\rangle$ in A. Since S and S^{-1} are positive elements in $b(L_A(E))$, we have

$$\bigl\langle \xi, S^{-1}(\xi)\bigr\rangle \leqslant \bigl\|S^{-1}\bigr\|_\infty \langle\xi,\xi\rangle$$

and

$$\langle\xi,\xi\rangle = \bigl\langle S\bigl(S^{-\frac{1}{2}}(\xi)\bigr), S^{-\frac{1}{2}}(\xi)\bigr\rangle \leqslant \|S\|_\infty \bigl\langle \xi, S^{-1}(\xi)\bigr\rangle$$

(see, for example, [7,11]). Therefore we have

$$\frac{1}{\|S\|_\infty}\langle\xi,\xi\rangle \leqslant \sum_n \langle\xi,h_n\rangle_{M(E)} \cdot \langle h_n,\xi\rangle_{M(E)} \leqslant \bigl\|S^{-1}\bigr\|_\infty \langle\xi,\xi\rangle$$

for each $\xi \in E$. $\square$

Definition 3.2. Let E be a Hilbert module over a pro-C^*-algebra A and $\{h_n\}_n$ a standard frame of multipliers for E. The invertible positive element $S \in b(L_A(E))$ such that $\sum_n h_n \cdot \langle h_n, S(\xi)\rangle_{M(E)} = \xi$ for all $\xi \in E$ is called the frame operator associated to the standard frame of multipliers $\{h_n\}_n$.

The following proposition improved a result of Lj. Arambašić [1, Theorem 2.5].

Proposition 3.3. *Let E and F be Hilbert modules over pro-C^*-algebra A and let $V : E \to F$ be a surjective module morphism. If $V \in b(L_A(E,F))$ and $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S, then $\{Vh_n\}_n$ is a standard frame of multipliers for F with the frame operator $(VS^{-1}V^*)^{-1}$.*

Proof. Since V is a surjective element in $b(L_A(E,F))$ and S is a positive invertible element in $b(L_A(E))$, $VS^{-\frac{1}{2}}$ is a surjective element in $b(L_A(E,F))$. Therefore $VS^{-1}V^*$ is a positive invertible element in $b(L_A(F))$ [7].

Let $\eta \in F$. From

$$\begin{aligned}
&\overline{p}_F\left(\sum_{k=1}^{n} Vh_k \cdot \langle Vh_k, (VS^{-1}V^*)^{-1}(\eta)\rangle_{M(F)} - \eta\right)\\
&\quad = \overline{p}_F\left(\sum_{k=1}^{n} V\left(h_k \cdot \langle Vh_k, (VS^{-1}V^*)^{-1}(\eta)\rangle_{M(F)}\right) - VS^{-1}V^*(VS^{-1}V^*)^{-1}(\eta)\right)\\
&\quad = \overline{p}_F\left(V\left(\sum_{k=1}^{n} h_k \cdot \langle h_k, V^*(VS^{-1}V^*)^{-1}(\eta)\rangle_{M(E)} - S^{-1}V^*(VS^{-1}V^*)^{-1}(\eta)\right)\right)\\
&\quad \leqslant \|V\|_\infty \overline{p}_E\left(\sum_{k=1}^{n} h_k \cdot \langle h_k, V^*(VS^{-1}V^*)^{-1}(\eta)\rangle_{M(E)} - S^{-1}V^*(VS^{-1}V^*)^{-1}(\eta)\right)
\end{aligned}$$

for all $p \in S(A)$ and for all positive integer n and taking into account that

$$\sum_n h_n \cdot \langle h_n, V^*(VS^{-1}V^*)^{-1}(\eta)\rangle_{M(E)} = S^{-1}V^*(VS^{-1}V^*)^{-1}(\eta)$$

(see, Proposition 3.1), we deduce that $\sum_n Vh_n \cdot \langle Vh_n, (VS^{-1}V^*)^{-1}(\eta)\rangle_{M(F)}$ converges to η. Therefore $\{Vh_n\}_n$ is a standard frame of multipliers for F with the frame operator $(VS^{-1}V^*)^{-1}$. □

Let E and F be Hilbert modules over pro-C^*-algebras A and B and let $\Phi : E \to F$ be an isomorphism of Hilbert modules such that the underlying pro-C^*-morphism φ is an isomorphism of pro-C^*-algebras. If $h \in M(E)$, then $\Phi \circ h \circ \varphi^{-1} \in M(F)$.

Indeed, from

$$\begin{aligned}
\langle \Phi \circ h \circ \varphi^{-1}(b), \eta\rangle &= \varphi\left(\langle h \circ \varphi^{-1}(b), \Phi^{-1}(\eta)\rangle\right) = \varphi\left(\langle \varphi^{-1}(b), h^*(\Phi^{-1}(\eta))\rangle\right)\\
&= \varphi\left(\varphi^{-1}(b^*)h^*(\Phi^{-1}(\eta))\right) = b^*\varphi\left(h^*(\Phi^{-1}(\eta))\right)\\
&= \langle b, \varphi(h^*(\Phi^{-1}(\eta)))\rangle = \langle b, \varphi \circ h^* \circ \Phi^{-1}(\eta)\rangle
\end{aligned}$$

for all $b \in B$ and for all $\eta \in F$, we deduce that $\Phi \circ h \circ \varphi^{-1} \in M(F)$, and moreover, $(\Phi \circ h \circ \varphi^{-1})^* = \varphi \circ h^* \circ \Phi^{-1}$.

Theorem 3.4. *Let E, F, Φ and φ be as above. If $\{h_n\}_n$ is a standard frame of multipliers for E, then $\{t_n\}_n$, where $t_n = \Phi \circ h_n \circ \varphi^{-1}$, is a standard frame of multipliers for F. Moreover if S is the frame operator associated to $\{h_n\}_n$, then $\Phi \circ S \circ \Phi^{-1}$ is the frame operator associated to $\{t_n\}_n$.*

Proof. To prove the theorem it is sufficient to show that $\Phi \circ S \circ \Phi^{-1}$ is an invertible positive element in $b(L_B(F))$ and the series $\sum_n t_n \cdot \langle t_n, \Phi \circ S \circ \Phi^{-1}(\eta)\rangle_{M(F)}$ converges to η in F for all $\eta \in F$. From

$$\begin{aligned}
\langle \Phi \circ S \circ \Phi^{-1}(\eta), \xi\rangle &= \varphi\left(\langle S(\Phi^{-1}(\eta)), \Phi^{-1}(\xi)\rangle\right)\\
&= \varphi\left(\langle \Phi^{-1}(\eta), S(\Phi^{-1}(\xi))\rangle\right) = \langle \eta, \Phi \circ S \circ \Phi^{-1}(\xi)\rangle
\end{aligned}$$

and

$$\langle \Phi \circ S \circ \Phi^{-1}(\eta), \eta\rangle = \varphi\left(\langle S(\Phi^{-1}(\eta)), \Phi^{-1}(\eta)\rangle\right) \geqslant 0$$

for all $\xi, \eta \in F$ we deduce that $\Phi \circ S \circ \Phi^{-1}$ is a positive element in $L_B(F)$. Clearly $\Phi \circ S \circ \Phi^{-1}$ is invertible, and since

$$\bar{q}_F\left(\Phi \circ S \circ \Phi^{-1}(\eta)\right)^2 = q\left(\varphi\left(\langle S \circ \Phi^{-1}(\eta), S \circ \Phi^{-1}(\eta)\rangle\right)\right)$$

(see, for example, [7])

$$\leqslant \|S\|_\infty^2 q\left(\varphi\left(\langle \Phi^{-1}(\eta), \Phi^{-1}(\eta)\rangle\right)\right) = \|S\|_\infty^2 \bar{q}_F(\eta)^2$$

for all $\eta \in F$ and for all $q \in S(B)$, $\Phi \circ S \circ \Phi^{-1}$ is an invertible positive element in $b(L_B(F))$.

Let $\eta \in F$, $q \in S(B)$. Since Φ is a morphism of Hilbert modules, there is $p \in S(A)$ such that $\bar{q}_F(\Phi(\xi)) \leqslant \bar{p}_E(\xi)$ for all $\xi \in E$. From

$$\begin{aligned}
&\bar{q}_F\left(\sum_{k=1}^{n} t_k \cdot \langle t_k, \Phi \circ S \circ \Phi^{-1}(\eta)\rangle_{M(F)} - \eta\right) \\
&= \bar{q}_F\left(\sum_{k=1}^{n} \Phi \circ h_k \circ \varphi^{-1} \cdot \langle \Phi \circ h_k \circ \varphi^{-1}, \Phi \circ S \circ \Phi^{-1}(\eta)\rangle_{M(F)} - \eta\right) \\
&= \bar{q}_F\left(\sum_{k=1}^{n} (\Phi \circ h_k \circ \varphi^{-1})\big((\varphi \circ h_k^* \circ \Phi^{-1})(\Phi \circ S \circ \Phi^{-1}(\eta))\big) - \eta\right) \\
&= \bar{q}_F\left(\Phi\left(\sum_{k=1}^{n} h_k\big(h_k^*\big(S\big(\Phi^{-1}(\eta)\big)\big)\big) - \Phi^{-1}(\eta)\right)\right) \\
&\leqslant \bar{p}_E\left(\sum_{k=1}^{n} h_k \cdot \langle h_k, S\big(\Phi^{-1}(\eta)\big)\rangle_{M(E)} - \Phi^{-1}(\eta)\right)
\end{aligned}$$

for all $n \in \mathbb{N}$, and taking into account that the series $\sum_k h_k \cdot \langle h_k, S(\Phi^{-1}(\eta))\rangle_{M(E)}$ converges to $\Phi^{-1}(\eta)$, we deduce that $\sum_n t_n \cdot \langle t_n, \Phi \circ S \circ \Phi^{-1}(\eta)\rangle_{M(F)}$ converges to η in F. Thus, we showed that $\{t_n\}_n$ is a standard frame of multipliers for F with the frame operator $\Phi \circ S \circ \Phi^{-1}$. $\square$

Corollary 3.5. *Let E and F be two Hilbert modules over a pro-C^*-algebra A. If $U : E \to F$ is a unitary operator and $\{h_n\}_n$ is a standard frame of multipliers for E with the operator frame S, then $\{U \circ h_n\}_n$ is a standard frame of multipliers for F with the operator frame USU^*.*

Proof. Since U is a unitary operator, it is an isomorphism of Hilbert modules with the underlying pro-C^*-morphism $\varphi = id_A$. $\square$

4. Frames of multipliers in the external tensor product of Hilbert modules

Let E be a Hilbert module over a pro-C^*-algebra A and F a Hilbert module over a pro-C^*-algebra B. The algebraic tensor product $E \otimes_{\text{alg}} F$ of E and F is a pre-Hilbert $A \otimes_{\min} B$-module with the action of $A \otimes_{\min} B$ on $E \otimes_{\text{alg}} F$ defined by

$$(\xi \otimes \eta)(a \otimes b) = \xi a \otimes \eta b$$

and the inner product defined by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2\rangle = \langle \xi_1, \xi_2\rangle \otimes \langle \eta_1, \eta_2\rangle.$$

The external tensor product of E and F is the Hilbert module $E \otimes F$ over $A \otimes_{\min} B$ obtained by the completion of the pre-Hilbert $A \otimes_{\min} B$-module $E \otimes_{\text{alg}} F$ [6].

If $h \in M(E)$ and $t \in M(F)$ then there is a unique adjointable module morphism $h \otimes t : A \otimes_{\min} B \to E \otimes F$ such that $(h \otimes t)(a \otimes b) = h(a) \otimes t(b)$ and $(h \otimes t)^*(a \otimes b) = h^*(a) \otimes t^*(b)$ for all $a \in A$ and for all $b \in B$ (see, for example, [9]).

Lemma 4.1. *Let E be a Hilbert A-module, E_0 a dense submodule of E, S a positive invertible element in $b(L_A(E))$ and $\{h_n\}_n$ a sequence of multipliers of E. If the series $\sum_n \langle \zeta, h_n\rangle_{M(E)}\langle h_n, \zeta\rangle_{M(E)}$ converges to $\langle \zeta, S^{-1}(\zeta)\rangle$ for all $\zeta \in E_0$, then $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S.*

Proof. To show that $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S it is sufficient to prove that the series $\sum_n \langle \zeta, h_n\rangle_{M(E)}\langle h_n, \zeta\rangle_{M(E)}$ converges to $\langle \zeta, S^{-1}(\zeta)\rangle$ for all $\zeta \in E$. Let $\zeta \in E$ and let $\{\zeta_i\}_{i\in I}$ be a net in E_0 which converges to ζ. Then, for any $p \in S(A)$ and for any $\varepsilon > 0$, there is $i_0 \in I$ such that

$$\bar{p}_E(\zeta_i - \zeta_{i_0}) < \sqrt{\frac{\varepsilon}{6\|S^{-1}\|_\infty}} \min\left\{1, \frac{\sqrt{\varepsilon}}{\sqrt{6\|S^{-1}\|_\infty}\bar{p}_E(\zeta)}\right\}$$

for all $i \in I$ with $i \geqslant i_0$, and since $\sum_n \langle \zeta_{i_0}, h_n\rangle_{M(E)}\langle h_n, \zeta_{i_0}\rangle_{M(E)}$ converges to $\langle \zeta_{i_0}, S^{-1}(\zeta_{i_0})\rangle$, there is a positive integer n_1 such that

$$p\left(\sum_{n=n_1}^{n_2} \langle \zeta_{i_0}, h_n\rangle_{M(E)}\langle h_n, \zeta_{i_0}\rangle_{M(E)}\right) < \frac{\varepsilon}{6}$$

and

$$p\left(\sum_{n=1}^{n_2}\langle\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_{i_0}\rangle_{M(E)}-\left\langle\zeta_{i_0},S^{-1}(\zeta_{i_0})\right\rangle\right)<\frac{\varepsilon}{6}$$

for all positive integer n_2 with $n_2\geqslant n_1$. Thus, we obtain

$$\begin{aligned}
p\left(\sum_{n=n_1}^{n_2}\langle\zeta,h_n\rangle_{M(E)}\langle h_n,\zeta\rangle_{M(E)}\right)&=\lim_i p\left(\sum_{n=n_1}^{n_2}\langle\zeta_i,h_n\rangle_{M(E)}\langle h_n,\zeta_i\rangle_{M(E)}\right)\\
&\leqslant\lim_i p\left(\sum_{n=n_1}^{n_2}\langle\zeta_i-\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_i-\zeta_{i_0}\rangle_{M(E)}\right)\\
&\quad+2\lim_i p\left(\sum_{n=n_1}^{n_2}\langle\zeta_i-\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_{i_0}\rangle_{M(E)}\right)\\
&\quad+p\left(\sum_{n=n_1}^{n_2}\langle\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_{i_0}\rangle_{M(E)}\right)\\
&\leqslant\lim_i p\left(\left\langle\zeta_i-\zeta_{i_0},S^{-1}(\zeta_i-\zeta_{i_0})\right\rangle\right)\\
&\quad+2\lim_i p\left(\sum_{n=n_1}^{n_2}\langle\zeta_i-\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_i-\zeta_{i_0}\rangle_{M(E)}\right)^{\frac{1}{2}}\\
&\quad\times p\left(\sum_{n=n_1}^{n_2}\langle\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_{i_0}\rangle_{M(E)}\right)^{\frac{1}{2}}\\
&\quad+p\left(\sum_{n=n_1}^{n_2}\langle\zeta_{i_0},h_n\rangle_{M(E)}\langle h_n,\zeta_{i_0}\rangle_{M(E)}\right)\\
&\leqslant\left\|S^{-1}\right\|_\infty\lim_i\overline{p}_E(\zeta_i-\zeta_{i_0})^2+2\lim_i p\left(\left\langle\zeta_i-\zeta_{i_0},S^{-1}(\zeta_i-\zeta_{i_0})\right\rangle\right)^{\frac{1}{2}}\sqrt{\frac{\varepsilon}{6}}+\frac{\varepsilon}{6}\\
&<\frac{\varepsilon}{6}+2\left\|S^{-1}\right\|_\infty^{\frac{1}{2}}\lim_i\overline{p}_E(\zeta_i-\zeta_{i_0})\sqrt{\frac{\varepsilon}{6}}+\frac{\varepsilon}{6}\\
&<\frac{\varepsilon}{6}+2\sqrt{\frac{\varepsilon}{6}}\sqrt{\frac{\varepsilon}{6}}+\frac{\varepsilon}{6}<\varepsilon
\end{aligned}$$

for all positive integer n_2 with $n_2\geqslant n_1$. Therefore, the series $\sum_n\langle\zeta,h_n\rangle_{M(E)}\langle h_n,\zeta\rangle_{M(E)}$ is convergent. We can also suppose that

$$\overline{p}_E(\zeta_i-\zeta_{i_0})\leqslant\frac{\varepsilon}{12M\sqrt{\|S^{-1}\|_\infty}}$$

where $M=p(\sum_n\langle\zeta,h_n\rangle_{M(E)}\langle h_n,\zeta\rangle_{M(E)})^{\frac{1}{2}}$, for all $i\in I$ with $i\geqslant i_0$. Then

$$\begin{aligned}
&p\left(\sum_{k=1}^{n}\langle\zeta,h_k\rangle_{M(E)}\langle h_k,\zeta\rangle_{M(E)}-\left\langle\zeta,S^{-1}(\zeta)\right\rangle\right)\\
&\quad\leqslant p\left(\sum_{k=1}^{n}\langle\zeta-\zeta_{i_0},h_k\rangle_{M(E)}\langle h_k,\zeta-\zeta_{i_0}\rangle_{M(E)}\right)+2p\left(\sum_{k=1}^{n}\langle\zeta-\zeta_{i_0},h_k\rangle_{M(E)}\langle h_k,\zeta\rangle_{M(E)}\right)\\
&\qquad+p\left(\sum_{k=1}^{n}\langle\zeta_{i_0},h_k\rangle_{M(E)}\langle h_k,\zeta_{i_0}\rangle_{M(E)}-\left\langle\zeta_{i_0},S^{-1}(\zeta_{i_0})\right\rangle\right)+p\left(\left\langle\zeta,S^{-1}(\zeta)\right\rangle-\left\langle\zeta_{i_0},S^{-1}(\zeta_{i_0})\right\rangle\right)\\
&\quad\leqslant\lim_i p\left(\sum_{k=1}^{n}\langle\zeta_i-\zeta_{i_0},h_k\rangle_{M(E)}\langle h_k,\zeta_i-\zeta_{i_0}\rangle_{M(E)}\right)\\
&\qquad+2\lim_i p\left(\sum_{k=1}^{n}\langle\zeta_i-\zeta_{i_0},h_k\rangle_{M(E)}\langle h_k,\zeta_i-\zeta_{i_0}\rangle_{M(E)}\right)^{\frac{1}{2}}p\left(\sum_{k=1}^{n}\langle\zeta,h_k\rangle_{M(E)}\langle h_k,\zeta\rangle_{M(E)}\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\frac{\varepsilon}{6}+\lim_i p\left(\left\langle \zeta_i-\zeta_{i_0}, S^{-1}(\zeta_i-\zeta_{i_0})\right\rangle\right)+2\lim_i p\left(\left\langle \zeta_i-\zeta_{i_0}, S^{-1}(\zeta)\right\rangle\right)$$
$$\leqslant \lim_i p\left(\sum_k \langle \zeta_i-\zeta_{i_0}, h_k\rangle_{M(E)}\langle h_k, \zeta_i-\zeta_{i_0}\rangle_{M(E)}\right)$$
$$+2\lim_i p\left(\sum_k \langle \zeta_i-\zeta_{i_0}, h_k\rangle_{M(E)}\langle h_k, \zeta-\zeta_{i_0}\rangle_{M(E)}\right)^{\frac{1}{2}} p\left(\sum_k \langle \zeta, h_k\rangle_{M(E)}\langle h_k, \zeta\rangle_{M(E)}\right)^{\frac{1}{2}}$$
$$+\frac{\varepsilon}{6}+\frac{\varepsilon}{6}+2\left\|S^{-1}\right\|_\infty \lim_i \overline{p}_E(\zeta_i-\zeta_{i_0})\overline{p}_E(\zeta)$$
$$\leqslant \lim_i p\left(\left\langle \xi_i-\zeta_{i_0}, S^{-1}(\xi_i-\zeta_{i_0})\right\rangle\right)+2\lim_i p\left(\left\langle \xi_i-\zeta_{i_0}, S^{-1}(\xi_i-\zeta_{i_0})\right\rangle\right)^{\frac{1}{2}} M+\frac{\varepsilon}{3}+2\frac{\varepsilon}{6}$$
$$<\frac{\varepsilon}{6}+2\frac{\varepsilon}{12}+\frac{\varepsilon}{3}+2\frac{\varepsilon}{6}=\varepsilon$$

for all positive integer n with $n \geqslant n_1$, and so

$$\sum_n \langle \zeta, h_n\rangle_{M(E)}\langle h_n, \zeta\rangle_{M(E)}=\left\langle \zeta, S^{-1}(\zeta)\right\rangle.$$

Thus we showed that $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S. □

Theorem 4.2. *Let $\{h_n\}_n$ and $\{t_m\}_m$ be standard frames of multipliers for E and F with the frame operators S_1 respectively S_2. Then $\{h_n \otimes t_m\}_{(n,m)}$ is a standard frame of multipliers for $E \otimes F$ with the frame operator $S_1 \otimes S_2$.*

Proof. Since S_1 and S_2 are positive invertible elements in $b(L_A(E))$ respectively $b(L_B(F))$, $S_1 \otimes S_2$ is a positive invertible element in $b(L_{A\otimes_{\min}B}(E \otimes F))$ with $(S_1 \otimes S_2)^{-1}=S_1^{-1}\otimes S_2^{-1}$, and then according to Lemma 4.1, to show that $\{h_n \otimes t_m\}_{(n,m)}$ is a standard frame of multipliers for $E \otimes F$ with the frame operator $S_1 \otimes S_2$ it is sufficient to show that the sequence

$$\left(\sum_{(i,j)=(1,1)}^{(n,m)} \left\langle (S_1\otimes S_2)(\zeta), h_i\otimes t_j\right\rangle_{M(E\otimes F)}\left\langle h_i\otimes t_j, (S_1\otimes S_2)(\zeta)\right\rangle_{M(E\otimes F)}\right)_{(n,m)}$$

converges to $\langle \zeta, (S_1^{-1}\otimes S_2^{-1})(\zeta)\rangle$ for all $\zeta \in E \otimes_{\mathrm{alg}} F$.

Let $\xi \in E$, $\eta \in F$, $p \in S(A)$ and $q \in S(B)$. From

$$\overline{p\otimes q}_{E\otimes F}\left(\sum_{(i,j)=(1,1)}^{(n,m)} (h_i\otimes t_j)\cdot\left\langle h_i\otimes t_j, (S_1\otimes S_2)(\xi\otimes\eta)\right\rangle_{M(E\otimes F)}-\xi\otimes\eta\right)$$
$$=\overline{p\otimes q}_{E\otimes F}\left(\sum_{i=1}^{n} h_i\cdot\left\langle h_i, S_1(\xi)\right\rangle_{M(E)}\otimes\sum_{j=1}^{m} t_j\cdot\left\langle t_j, S_2(\eta)\right\rangle_{M(F)}-\xi\otimes\eta\right)$$
$$\leqslant \overline{p}_E\left(\sum_{i=1}^{n} h_i\cdot\left\langle h_i, S_1(\xi)\right\rangle_{M(E)}-\xi\right)\overline{q}_F\left(\sum_{j=1}^{m} t_j\cdot\left\langle t_j, S_2(\eta)\right\rangle_{M(F)}-\eta\right)$$
$$+\overline{p}_E(\xi)\overline{q}_F\left(\sum_{j=1}^{m} t_j\cdot\left\langle t_j, S_2(\eta)\right\rangle_{M(F)}-\eta\right)+\overline{p}_E\left(\sum_{i=1}^{n} h_i\cdot\left\langle h_i, S_1(\xi)\right\rangle_{M(E)}-\xi\right)\overline{q}_F(\eta)$$

and taking into account that $\xi=\sum_n h_n\cdot\langle h_n, S_1(\xi)\rangle_{M(E)}$ and $\eta=\sum_m t_m\cdot\langle t_m, S_2(\eta)\rangle_{M(F)}$ we deduce that $(\sum_{(i,j)=(1,1)}^{(n,m)}(h_i\otimes t_j)\cdot\langle h_i\otimes t_j, (S_1\otimes S_2)(\xi\otimes\eta)\rangle_{M(E\otimes F)})_{(n,m)}$ converges to $\xi\otimes\eta$.

Let $\zeta=\sum_{k=1}^{l}\xi_k\otimes\eta_k$. Then

$$\zeta=\sum_{k=1}^{l}\left(\sum_{(n,m)}(h_n\otimes t_m)\cdot\left\langle h_n\otimes t_m, (S_1\otimes S_2)(\xi_k\otimes\eta_k)\right\rangle_{M(E\otimes F)}\right)$$
$$=\sum_{(n,m)}(h_n\otimes t_m)\cdot\left\langle h_n\otimes t_m, (S_1\otimes S_2)(\zeta)\right\rangle_{M(E\otimes F)}.$$

From this relation, we deduce that $\sum_{(n,m)}\langle \zeta, h_n \otimes t_m\rangle_{M(E\otimes F)}\langle h_n \otimes t_m, \zeta\rangle_{M(E\otimes F)}$ converges in $A \otimes_{\min} B$ for each $\zeta \in E \otimes_{\mathrm{alg}} F$, and moreover,

$$\big\langle \zeta, \big(S_1^{-1} \otimes S_2^{-1}\big)(\zeta)\big\rangle = \sum_{(n,m)} \langle \zeta, h_n \otimes t_m\rangle_{M(E\otimes F)}\langle h_n \otimes t_m, \zeta\rangle_{M(E\otimes F)}.$$

Therefore $\{h_n \otimes t_m\}_{(n,m)}$ is a standard frame for $E \otimes F$ with the frame operator $S_1 \otimes S_2$. $\square$

In the following proposition we show that given a standard frame of multipliers for a Hilbert A-module E, if the Hilbert B-module F has a multiplier which verifies some conditions, then we can construct a standard frame of multipliers for the external tensor product of E and F.

Proposition 4.3. *Let E be a Hilbert A-module, F a Hilbert B-module. If $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S and t is a multiplier of F such that tt^* is an invertible element in $b(L_B(F))$, then $\{h_n \otimes t\}_n$ is a standard frame of multipliers for $E \otimes F$ with the frame operator $S \otimes (tt^*)^{-1}$.*

Proof. Indeed, from

$$\begin{aligned}
&p \otimes q\left(\sum_{k=1}^{n}\langle \xi \otimes \eta, h_k \otimes t\rangle_{M(E\otimes F)}\langle h_k \otimes t, \xi \otimes \eta\rangle_{M(E\otimes F)} - \big\langle \xi \otimes \eta, \big(S \otimes \big(tt^*\big)^{-1}\big)^{-1}(\xi \otimes \eta)\big\rangle\right)\\
&\quad = p \otimes q\left(\sum_{k=1}^{n}\langle \xi, h_k\rangle_{M(E)}\langle h_k, \xi\rangle_{M(E)} \otimes \langle \eta, t\rangle_{M(F)}\langle t, \eta\rangle_{M(F)} - \big\langle \xi \otimes \eta, S^{-1}(\xi) \otimes tt^*(\eta)\big\rangle\right)\\
&\quad = p \otimes q\left(\sum_{k=1}^{n}\langle \xi, h_k\rangle_{M(E)}\langle h_k, \xi\rangle_{M(E)} \otimes \big\langle \eta, tt^*(\eta)\big\rangle - \big\langle \xi, S^{-1}(\xi)\big\rangle \otimes \big\langle \eta, tt^*(\eta)\big\rangle\right)\\
&\quad = p \otimes q\left(\left(\sum_{k=1}^{n}\langle \xi, h_k\rangle_{M(E)}\langle h_k, \xi\rangle_{M(E)} - \big\langle \xi, S^{-1}(\xi)\big\rangle\right) \otimes \big\langle \eta, tt^*(\eta)\big\rangle\right)\\
&\quad \leqslant p\left(\sum_{k=1}^{n}\langle \xi, h_k\rangle_{M(E)}\langle h_k, \xi\rangle_{M(E)} - \big\langle \xi, S^{-1}(\xi)\big\rangle\right) q\big(\big\langle \eta, tt^*(\eta)\big\rangle\big)
\end{aligned}$$

for all $p \in S(A)$, for all $q \in S(B)$, for all $\xi \in E$, for all $\eta \in F$ and for all positive integer n, and taking into account that $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S, we deduce that $\sum_n \langle \zeta, h_n \otimes t\rangle_{M(E\otimes F)}\langle h_n \otimes t, \zeta\rangle_{M(E\otimes F)}$ converges to $\langle \zeta, (S \otimes (tt^*)^{-1})^{-1}(\zeta)\rangle$ for all $\zeta \in E \otimes_{\mathrm{alg}} F$ and by Lemma 4.1 $\{h_n \otimes t\}_n$ is a standard frame of multipliers for $E \otimes F$ with the frame operator $S \otimes (tt^*)^{-1}$. $\square$

Remark 4.4. Given a standard frame of multipliers for the Hilbert $A \otimes_{\min} B$-module $E \otimes F$ it is not easy to construct a standard frame of multipliers for E or F because, in general, the pro-C^*-algebras $A \otimes_{\min} B$ and A respectively B are not isomorphic, and so we cannot construct an isomorphism of Hilbert modules from $E \otimes F$ to E respectively F.

Example 4.5. Suppose that A is a stable pro-C^*-algebra (that is, pro-C^*-algebras $A \otimes \mathcal{K}$ and A, where $\mathcal{K}$ is the C^*-algebra of all compact operators on an infinite dimensional separable Hilbert space H). Let $\varphi : A \otimes \mathcal{K} \to A$ be a pro-C^*-isomorphism. Then the linear map $\Psi : H \otimes_{\mathrm{alg}} (A \otimes \mathcal{K}) \to H \otimes_{\mathrm{alg}} A$ defined by

$$\Psi(\xi \otimes x) = \xi \otimes \varphi(x)$$

extended to an isomorphism of Hilbert modules from $H \otimes (A \otimes \mathcal{K}) \to H \otimes A$, since

$$\begin{aligned}
\big\langle \Psi(\xi \otimes x), \Psi(\xi \otimes x)\big\rangle &= \big\langle \xi \otimes \varphi(x), \xi \otimes \varphi(x)\big\rangle = \langle \xi, \xi\rangle \otimes \big\langle \varphi(x), \varphi(x)\big\rangle\\
&= \langle \xi, \xi\rangle \varphi(x)^* \varphi(x) = \varphi\big(\langle \xi, \xi\rangle x^* x\big) = \varphi\big(\langle \xi, \xi\rangle \otimes \langle x, x\rangle\big) = \varphi\big(\langle \xi \otimes x, \xi \otimes x\rangle\big)
\end{aligned}$$

for all $\xi \in H$ and for all $x \in A \otimes \mathcal{K}$ and since φ is a pro-C^*-isomorphism.

On the other hand the Hilbert modules $H \otimes A$ and H_A are unitarily equivalent as well as the Hilbert modules $H \otimes A \otimes \mathcal{K}$ and $H_A \otimes \mathcal{K}$ (see, for example, [7]). Therefore there is a unitary operator $U : H_A \to H \otimes A$ and a unitary operator $V : H \otimes A \otimes \mathcal{K} \to H_A \otimes \mathcal{K}$.

Let $\Phi = U \circ \Psi \circ V$. Then $\Phi : H_A \otimes \mathcal{K} \to H_A$ is an isomorphism of Hilbert modules with the underlying pro-C^*-isomorphism φ and if $\{h_n\}_n$ is a standard frame of multipliers for $H_A \otimes \mathcal{K}$ with the frame operator S then $\{\Phi \circ h_n \circ \varphi^{-1}\}_n$ is a standard frame of multipliers for H_A with the frame operator $\Phi \circ S \circ \Phi^{-1}$ (see, Theorem 3.4).

5. Frames of multipliers in the inner tensor product of Hilbert modules

Let E and F be Hilbert modules over pro-C^*-algebras A and B and let $\varphi : A \to L_B(F)$ be a non-degenerate pro-C^*-morphism (that is, $\varphi(A)F$ is dense in F). The algebraic tensor product $E \otimes_A F$ of E and F over A is a pre-Hilbert B-module with the action of B on $E \otimes_A F$ defined by

$$(\xi \otimes_A \eta)b = \xi \otimes_A \eta b$$

and the inner product defined by

$$\langle \xi_1 \otimes_A \eta_1, \xi_2 \otimes_A \eta_2 \rangle = \big\langle \eta_1, \varphi\big(\langle \xi_1, \xi_2 \rangle\big)\eta_2 \big\rangle.$$

The inner tensor product of E and F using φ is the Hilbert module $E \otimes_\varphi F$ over B obtained by the completion of the pre-Hilbert B-module $E \otimes_A F$ [6]. Moreover, there is a unique pro-C^*-morphism $\varphi_* : L_A(E) \to L_B(E \otimes_\varphi F)$ such that $\varphi_*(T)(\xi \otimes_\varphi \eta) = T(\xi) \otimes_\varphi \eta$.

For $h \in M(E)$ and $t \in M(F)$, the map $h \otimes_\varphi t : B \to E \otimes_\varphi F$ defined by

$$(h \otimes_\varphi t)(b) = \lim_i \big(h(e_i) \otimes_\varphi t(b)\big),$$

where $\{e_i\}_{i \in I}$ is an approximate unity for A, defines an element in $M(E \otimes_\varphi F)$ (see, for example, [9]). Moreover, $(h \otimes_\Phi t)^*(\xi \otimes_\varphi \eta) = t^*(\varphi(h^*(\xi))\eta)$.

Remark 5.1. Let E, F, φ, h and t be as above. Suppose that $tt^* \in \varphi(h^*(E))'$, the commutant of $\varphi(h^*(E))$ in $L_B(F)$. Then

$$\begin{aligned}
&(h \otimes_\varphi t) \cdot \langle h \otimes_\varphi t, \xi \otimes_\varphi \eta \rangle_{M(E \otimes_\Phi F)} \\
&\quad = (h \otimes_\varphi t) \cdot t^*\big(\varphi\big(h^*(\xi)\big)\eta\big) = (h \otimes_\varphi t)\big(t^*\big(\varphi\big(h^*(\xi)\big)\eta\big)\big) \\
&\quad = \lim_i \big(h(e_i) \otimes_\varphi t\big(t^*\big(\varphi\big(h^*(\xi)\big)\eta\big)\big)\big) = \lim_i \big(h(e_i) \otimes_\varphi tt^*\big(\varphi\big(h^*(\xi)\big)\eta\big)\big) \\
&\quad = \lim_i \big(h(e_i) \otimes_\varphi \varphi\big(h^*(\xi)\big)\big(tt^*(\eta)\big)\big) = \lim_i \big(h(e_i)h^*(\xi) \otimes_\varphi tt^*(\eta)\big) \\
&\quad = \lim_i \big(h\big(e_i h^*(\xi)\big) \otimes_\varphi tt^*(\eta)\big) = h\big(h^*(\xi)\big) \otimes_\varphi tt^*(\eta) \\
&\quad = h \cdot \langle h, \xi \rangle_{M(E)} \otimes_\varphi t \cdot \langle t, \eta \rangle_{M(F)}
\end{aligned}$$

for all $\xi \in E$ and for all $\eta \in F$.

For a Hilbert module E over A, $\langle E, E \rangle$ denotes the closed two-sided $*$-ideal of A generated by $\{\langle \xi, \eta \rangle;\ \xi, \eta \in E\}$.

Lemma 5.2. *Let E, F and φ be as above. If $S_1 \in b(L_A(E))$ and $S_2 \in b(L_B(F)) \cap \varphi(\langle E, E \rangle)'$, then there is $S_1 \otimes_\varphi S_2 \in b(L_B(E \otimes_\varphi F))$ such that*

$$(S_1 \otimes_\varphi S_2)(\xi \otimes_\varphi \eta) = S_1\xi \otimes_\varphi S_2\eta.$$

Moreover, if S_1 and S_2 are positive (invertible) adjointable module morphisms, then $S_1 \otimes_\varphi S_2$ is a positive (invertible) module morphism.

Proof. Consider the map $S_1 \otimes_\varphi S_2 : E \otimes_A F \to E \otimes_A F$ defined by

$$(S_1 \otimes_\varphi S_2)\left(\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k\right) = \sum_{k=1}^n S_1(\xi_k) \otimes_\varphi S_2(\eta_k).$$

Since

$$\begin{aligned}
&\bar{q}_F\left((S_1 \otimes_\varphi S_2)\left(\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k\right)\right)^2 \\
&\quad = q\left(\left\langle \sum_{k=1}^n S_1(\xi_k) \otimes_\varphi S_2(\eta_k), \sum_{k=1}^n S_1(\xi_k) \otimes_\varphi S_2(\eta_k) \right\rangle\right) \\
&\quad = q\left(\left\langle \sum_{k=1}^n \varphi_*(S_1)\big(\xi_k \otimes_\varphi S_2(\eta_k)\big), \sum_{k=1}^n \varphi_*(S_1)\big(\xi_k \otimes_\varphi S_2(\eta_k)\big) \right\rangle\right)
\end{aligned}$$

$$= q\left(\left\langle \varphi_*\left(S_1^* S_1\right)\left(\sum_{k=1}^n \xi_k \otimes_\varphi S_2(\eta_k)\right), \left(\sum_{k=1}^n \xi_k \otimes_\varphi S_2(\eta_k)\right)\right\rangle\right)$$
$$\leqslant \|S_1\|_\infty^2 q\left(\left\langle \sum_{k=1}^n \xi_k \otimes_\varphi S_2(\eta_k), \sum_{k=1}^n \xi_k \otimes_\varphi S_2(\eta_k)\right\rangle\right)$$
$$\leqslant \|S_1\|_\infty^2 q\left(\sum_{k,l=1}^n \left\langle S_2(\eta_k), \varphi\left(\langle \xi_k, \xi_l\rangle\right) S_2(\eta_l)\right\rangle\right)$$
$$= \|S_1\|_\infty^2 q\left(\left\langle \left[\delta_{kl} S_2^* S_2\right]_{k,l=1}^n \left(\left[\varphi\left(\langle \xi_k, \xi_l\rangle\right)\right]_{k,l=1}^n\right)^{\frac{1}{2}} (\eta_k)_{k=1}^n, \left(\left[\varphi\left(\langle \xi_k, \xi_l\rangle\right)\right]_{k,l=1}^n\right)^{\frac{1}{2}} (\eta_l)_{l=1}^n\right\rangle\right)$$
$$\leqslant \|S_1\|_\infty^2 \|S_2\|_\infty^2 q\left(\left\langle (\eta_k)_{k=1}^n, \left[\varphi\left(\langle \xi_k, \xi_l\rangle\right)\right]_{k,l=1}^n (\eta_l)_{l=1}^n\right\rangle\right)$$
$$= \|S_1\|_\infty^2 \|S_2\|_\infty^2 \bar{q}_F\left(\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k\right)^2$$

for all $q \in S(B)$ and for all $\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k \in E \otimes_A F$, $S_1 \otimes_\varphi S_2$ extends by continuity to a linear map, denoted also by $S_1 \otimes_\varphi S_2$, from $E \otimes_\varphi F$ to $E \otimes_\varphi F$ and $\bar{q}_F((S_1 \otimes_\varphi S_2)(\zeta)) \leqslant \|S_1\|_\infty \|S_2\|_\infty \bar{q}_F(\zeta)$ for all $\zeta \in E \otimes_\varphi F$ and for all $q \in S(B)$.

From

$$\left\langle (S_1 \otimes_\varphi S_2)\left(\sum_{k=1}^n \xi_k^1 \otimes_\varphi \eta_k^1\right), \sum_{k=1}^m \xi_k^2 \otimes_\varphi \eta_k^2\right\rangle = \sum_{k,l=1}^{n,m} \left\langle S_2\left(\eta_k^1\right), \varphi\left(\left\langle S_1\left(\xi_k^1\right), \xi_l^2\right\rangle\right)\eta_l^2\right\rangle$$
$$= \sum_{k,l=1}^{n,m} \left\langle \eta_k^1, \varphi\left(\left\langle \xi_k^1, S_1\left(\xi_l^2\right)\right\rangle\right) S_2\left(\eta_l^2\right)\right\rangle$$
$$= \left\langle \sum_{k=1}^n \xi_k^1 \otimes_\varphi \eta_k^1, \sum_{k=1}^m S_1\left(\xi_k^2\right) \otimes_\varphi S_2\left(\eta_k^2\right)\right\rangle$$
$$= \left\langle \left(\sum_{k=1}^n \xi_k^1 \otimes_\varphi \eta_k^1\right), (S_1 \otimes_\varphi S_2) \sum_{k=1}^m \xi_k^2 \otimes_\varphi \eta_k^2\right\rangle$$

and

$$\left\langle (S_1 \otimes_\varphi S_2)\left(\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k\right), \sum_{k=1}^n \xi_k \otimes_\varphi \eta_k\right\rangle = \sum_{k,l=1}^n \left\langle S_2(\eta_k), \varphi\left(\left\langle S_1(\xi_k), \xi_l\right\rangle\right)\eta_l\right\rangle$$
$$= \left\langle \left(|S_2|\eta_k\right)_{k=1}^n, \left[\varphi\left(\left\langle S_1(\xi_k), \xi_l\right\rangle\right)\right]_{k,l=1}^n \left(|S_2|\eta_l\right)_{l=1}^n\right\rangle \geqslant 0$$

for all $\sum_{k=1}^n \xi_k^1 \otimes_\varphi \eta_k^1, \sum_{k=1}^m \xi_k^2 \otimes_\varphi \eta_k^2 \in E \otimes_\varphi F$ and for all $\sum_{k=1}^n \xi_k \otimes_\varphi \eta_k \in E \otimes_\varphi F$, we deduce that $S_1 \otimes_\varphi S_2$ is a positive element in $L_B(E \otimes_\varphi F)$. Moreover, if S_1 and S_2 are invertible, since $(S_1 \otimes_\varphi S_2)((S_1^{-1} \otimes_\varphi S_2^{-1})(\xi \otimes_\varphi \eta)) = \xi \otimes_\varphi \eta = (S_1^{-1} \otimes_\varphi S_2^{-1})((S_1 \otimes_\varphi S_2)(\xi \otimes_\varphi \eta))$ for all $\xi \otimes_\varphi \eta \in E \otimes_\varphi F$, $S_1 \otimes_\varphi S_2$ is invertible. □

Theorem 5.3. *Let E, F and φ be as above. If $\{h_n\}_n$ is a standard frame of multipliers for E with the frame operator S_1 and $\{t_m\}_m$ is a standard frame of multipliers for F such that $t_m t_m^* \in \varphi(\langle E, E\rangle)'$ with the frame operator S_2, then $\{h_n \otimes_\varphi t_m\}_{(n,m)}$ is a standard frame for $E \otimes_\varphi F$ with the frame operator $S_1 \otimes_\varphi S_2$.*

Proof. From

$$\varphi\left(\langle \xi, \eta\rangle\right)\left(S_2^{-1}(\zeta)\right) = \sum_m \varphi\left(\langle \xi, \eta\rangle\right)\left(t_m \cdot \langle t_m, \zeta\rangle_{M(F)}\right) = \sum_m \left(\varphi\left(\langle \xi, \eta\rangle\right) t_m t_m^*\right)(\zeta)$$
$$= \sum_m \left(t_m t_m^* \varphi\left(\langle \xi, \eta\rangle\right)\right)(\zeta) = \sum_m t_m \cdot \left\langle t_m, \varphi\left(\langle \xi, \eta\rangle\right)(\zeta)\right\rangle_{M(F)} = S_2^{-1}\left(\varphi\left(\langle \xi, \eta\rangle\right)(\zeta)\right)$$

for all $\xi, \eta, \zeta \in E$, we deduce that $S_2 \in \varphi(\langle E, E\rangle)'$, and by Lemma 5.2, $S_1 \otimes_\varphi S_2$ is a positive invertible element in $b(L_B(E \otimes_\Phi F))$.

Let $\xi \in E$, $\eta \in F$ and $q \in S(B)$. From

$$\begin{aligned}
&\bar{q}_{E\otimes_\varphi F}\left(\sum_{(k,l)=(1,1)}^{(n,m)} (h_k \otimes_\varphi t_l)\cdot \big\langle h_k \otimes_\varphi t_l, (S_1 \otimes_\varphi S_2)(\xi \otimes_\varphi \eta)\big\rangle_{M(E\otimes_\Phi F)} - \xi \otimes_\varphi \eta\right) \\
&= \bar{q}_{E\otimes_\varphi F}\left(\sum_{k=1}^{n} h_k \cdot \big\langle h_k, S_1(\xi)\big\rangle_{M(E)} \otimes_\varphi \left(\sum_{l=1}^{m} t_l \cdot \big\langle t_l, S_2(\eta)\big\rangle_{M(F)}\right) - \xi \otimes_\varphi \eta\right) \\
&\leqslant \bar{q}_{E\otimes_\varphi F}\left(\left(\sum_{k=1}^{n} h_k \cdot \big\langle h_k, S_1(\xi)\big\rangle_{M(E)} - \xi\right) \otimes_\varphi \left(\sum_{l=1}^{m} t_l \cdot \big\langle t_l, S_2(\eta)\big\rangle_{M(F)} - \eta\right)\right) \\
&\quad + \bar{q}_{E\otimes_\varphi F}\left(\left(\sum_{k=1}^{n} h_k \cdot \big\langle h_k, S_1(\xi)\big\rangle_{M(E)} - \xi\right) \otimes_\varphi \eta\right) \\
&\quad + \bar{q}_{E\otimes_\varphi F}\left(\xi \otimes_\varphi \left(\sum_{l=1}^{m} t_l \cdot \big\langle t_l, S_2(\eta)\big\rangle_{M(F)} - \eta\right)\right) \\
&\leqslant \bar{p}_E\left(\sum_{k=1}^{n} h_k \cdot \big\langle h_k, S_1(\xi)\big\rangle_{M(E)} - \xi\right)\bar{q}_F\left(\sum_{l=1}^{m} t_l \cdot \big\langle t_l, S_2(\eta)\big\rangle_{M(F)} - \eta\right) \\
&\quad + \bar{p}_E\left(\sum_{k=1}^{n} h_k \cdot \big\langle h_k, S_1(\xi)\big\rangle_{M(E)} - \xi\right)\bar{q}_F(\eta) \\
&\quad + \bar{p}_E(\xi)\bar{q}_F\left(\sum_{l=1}^{m} t_l \cdot \big\langle t_l, (S_2)\eta\big\rangle_{M(F)} - \eta\right)
\end{aligned}$$

for all positive integers n and m, and taking into account that $\sum_n h_n \cdot \langle h_n, S_2(\xi)\rangle_{M(E)}$ converges to ξ in E and $\sum_m t_m \cdot \langle t_m, S_2(\eta)\rangle_{M(F)}$ coverages to η in F, we deduce that $\sum_{(n,m)}(h_n \otimes_\varphi t_m)\cdot \langle h_n \otimes_\varphi t_m, (S_1 \otimes_\varphi S_2)(\xi \otimes_\varphi \eta)\rangle_{M(E\otimes_\varphi F)}$ converges to $\xi \otimes_\varphi \eta$ in $E \otimes_\varphi F$. Then $\sum_{(n,m)}(h_n \otimes_\varphi t_m)\cdot \langle h_n \otimes_\varphi t_m, (S_1^{-1} \otimes_\varphi S_2^{-1})(\zeta)\rangle_{M(E\otimes_\varphi F)}$ converges to ζ in $E \otimes_\varphi F$ for all $\zeta = \sum_{l=1}^{k} \xi_l \otimes_\varphi \eta_l$. Therefore

$$\big\langle \zeta, \big(S_1^{-1} \otimes_\varphi S_2^{-1}\big)(\zeta)\big\rangle = \sum_{(n,m)} \langle \zeta, h_n \otimes_\varphi t_m\rangle_{M(E\otimes F)}\langle h_n \otimes_\varphi t_m, \zeta\rangle_{M(E\otimes_\varphi F)}$$

for all $\zeta = \sum_{l=1}^{k} \xi_l \otimes_\varphi \eta_l$ and by Lemma 4.1, $\{h_n \otimes_\varphi t_m\}_{(n,m)}$ is a standard frame of multipliers for $E \otimes_\varphi F$ with the frame operator
$S_1 \otimes_\varphi S_2$. $\square$

Example 5.4. Let $\varphi : A \to B$ be a non-degenerate morphism of pro-C^*-algebras. Then the map $\widetilde{\varphi} : A \to L_B(H_B)$ defined by $\widetilde{\varphi}(a)((b_m)_m) = (\varphi(a)b_m)_m$ is a non-degenerate morphism of pro-C^*-algebras.

Let $\{e_n\}_n$ be the normalized standard frame of multipliers for H_B, where $e_n(b) = (\delta_{mn}b)_m$. Since

$$\big(\big(e_n e_n^*\big)\widetilde{\varphi}(a)\big)\big((b_m)_m\big) = \big(\delta_{mn}\varphi(a)b_m\big)_m = \big(\widetilde{\varphi}(a)\big(e_n e_n^*\big)\big)\big((b_m)_m\big)$$

for all $a \in A$, for all $(b_m)_m \in H_B$ and for all n, $\{e_n\}_n \subset \widetilde{\varphi}(A)'$.

Let E be a Hilbert A-module and $\{h_n\}_n$ a standard frame of multipliers for E with the frame operator S. Then, by Theorem 5.3, $\{h_n \otimes_{\widetilde{\varphi}} e_m\}_{(n,m)}$ is a standard frame of multipliers for $E \otimes_{\widetilde{\varphi}} H_B$ with the frame operator $(\widetilde{\varphi})_*(S)$.

We have seen that given a standard frame of multipliers for E and a standard frame of multipliers for F, we can construct a standard frame of multipliers for $E \otimes_\varphi F$. Conversely, given a standard frame of multipliers for $E \otimes_\varphi F$, can we construct a standard frame of multipliers for E or F?

Proposition 5.5. *Let E and F be Hilbert modules over pro-C^*-algebras A and B, let $\varphi : A \to L_B(F)$ be a non-degenerate pro-C^*-morphism and let $\{h_n\}_n$ be a standard frame of multipliers for $E \otimes_\varphi F$. Suppose that there is an element $\xi_0 \in b(E)$ such that $\varphi(\langle \xi_0, \xi_0\rangle)$ is surjective. Then there is an element $V_{\xi_0} \in L_B(E \otimes_\varphi F, F)$ such that $\{V_{\xi_0}h_n\}_n$ is a standard frame of multipliers for F. Moreover, the frame bounds are $\|(V_{\xi_0}S^{-1}(V_{\xi_0})^*)^{-1}\|_\infty^{-1}$ and $\|V_{\xi_0}S^{-1}(V_{\xi_0})^*\|_\infty$, where S is the frame operator associated to $\{h_n\}_n$.*

Proof. Since

$$\begin{aligned}\bar{q}_F\left(\sum_{k=1}^{n}\varphi\big(\langle\xi_0,\xi_k\rangle\big)\eta_k\right) &= \sup\left\{q\left(\left\langle\sum_{k=1}^{n}\varphi\big(\langle\xi_0,\xi_k\rangle\big)\eta_k,\eta\right\rangle\right);\ \bar{q}_F(\eta)\leqslant 1\right\}\\ &= \sup\left\{q\left(\left\langle\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k,\xi_0\otimes_\varphi\eta\right\rangle\right);\ \bar{q}_F(\eta)\leqslant 1\right\}\\ &\leqslant \sup\left\{\bar{q}_{E\otimes_\varphi F}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)\bar{q}_{E\otimes_\varphi F}(\xi_0\otimes_\varphi\eta);\ \bar{q}_F(\eta)\leqslant 1\right\}\\ &\leqslant \sup\left\{\bar{q}_{E\otimes_\varphi F}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)\bar{q}_F(\eta)\overline{p}_E(\xi_0);\ \bar{q}_F(\eta)\leqslant 1\right\}\\ &= \bar{q}_{E\otimes_\varphi F}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)\overline{p}_E(\xi_0)\leqslant\bar{q}_{E\otimes_\varphi F}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)\|\xi_0\|_\infty\end{aligned}$$

for all $\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\in E\otimes_\varphi F$ and for all $q\in S(B)$, there is a continuous linear map $V_{\xi_0}:E\otimes_\varphi F\to F$ such that

$$V_{\xi_0}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)=\sum_{k=1}^{n}\varphi\big(\langle\xi_0,\xi_k\rangle\big)\eta_k.$$

Consider the map $V_0:F\to E\otimes_\varphi F$ defined by $V_0\eta=\xi_0\otimes_\varphi\eta$. From

$$\begin{aligned}\left\langle\eta,V_{\xi_0}\left(\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right)\right\rangle &= \left\langle\eta,\sum_{k=1}^{n}\varphi\big(\langle\xi_0,\xi_k\rangle\big)\eta_k\right\rangle=\sum_{k=1}^{n}\langle\xi_0\otimes_\varphi\eta,\xi_k\otimes_\varphi\eta_k\rangle\\ &= \left\langle\xi_0\otimes_\varphi\eta,\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right\rangle=\left\langle V_0\eta,\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\right\rangle\end{aligned}$$

for all $\sum_{k=1}^{n}\xi_k\otimes_\varphi\eta_k\in E\otimes_\varphi F$ and for all $\eta\in F$, we deduce that V_{ξ_0} is adjointable and moreover, $(V_{\xi_0})^*=V_0$ and $\widetilde{q}_{L_B(F,E\otimes_\varphi F)}(V_{\xi_0})\leqslant\|\xi_0\|_\infty$ for all $q\in S(B)$. Moreover, V_{ξ_0} is surjective, since $\varphi(\langle\xi_0,\xi_0\rangle)$ is surjective and then for any $\eta\in F$ there is $\widetilde{\eta}\in F$ such that $\eta=\varphi(\langle\xi_0,\xi_0\rangle)\widetilde{\eta}=V_{\xi_0}(\xi_0\otimes_\varphi\widetilde{\eta})$. Therefore, V_{ξ_0} is a surjective bounded operator from F to $E\otimes_\varphi F$ and then by Proposition 3.3, $\{V_{\xi_0}h_n\}_n$ is a standard frames of multipliers for F with the frame operator $(V_{\xi_0}S^{-1}(V_{\xi_0})^*)^{-1}$. □

Proposition 5.6. *Let $\varphi:A\to B$ be an isomorphism of pro-C^*-algebras and E a Hilbert module over A. Then the Hilbert modules E and $E\otimes_\varphi B$ are isomorphic.*

Proof. Let $\{e_i\}_{i\in I}$ be an approximate unit for A and $\xi\in E$. From

$$\begin{aligned}\bar{q}_{E\otimes_\varphi B}(\xi\otimes_\varphi e_j-\xi\otimes_\varphi e_i)^2 &= q\big(\langle\xi\otimes_\varphi(e_j-e_i),\xi\otimes_\varphi(e_j-e_i)\rangle\big)\\ &= q\big((e_j-e_i)\varphi\big(\langle\xi,\xi\rangle\big)(e_j-e_i)\big)\leqslant 2q\big(\varphi\big(\langle\xi,\xi\rangle\big)e_j-\varphi\big(\langle\xi,\xi\rangle\big)e_i\big)\end{aligned}$$

for all $q\in S(B)$ and taking into account that $\{\varphi(\langle\xi,\xi\rangle)e_i\}_{i\in I}$ is a Cauchy net, we deduce that $\{\xi\otimes_\varphi e_i\}_{i\in I}$ is a Cauchy net and so it is convergent. Thus we can define a map $\Phi:E\to E\otimes_\varphi B$ by $\Phi(\xi)=\lim_i\xi\otimes_\varphi e_i$. Since

$$\big\langle\Phi(\xi),\Phi(\xi)\big\rangle=\lim_i\Big(\lim_j\langle\xi\otimes_\varphi e_i,\xi\otimes_\varphi e_j\rangle\Big)=\lim_i\Big(\lim_j e_i\varphi\big(\langle\xi,\xi\rangle e_j\big)\Big)=\varphi\big(\langle\xi,\xi\rangle\big)$$

for all $\xi\in E$, and since φ is an isomorphism of pro-C^*-algebras, to show that Φ is an isomorphism of Hilbert modules it remains to show that Φ is surjective. Let $p\in S(A)$. Since φ^{-1} is a morphism of pro-C^*-algebras, there is $q\in S(B)$ such that $p(\varphi^{-1}(b))\leqslant q(b)$ for all $b\in B$. Then

$$\begin{aligned}\overline{p}_{E\otimes_\varphi B}(\xi)^2 &= p\big(\langle\xi,\xi\rangle\big)=p\big(\varphi^{-1}\big(\varphi\big(\langle\xi,\xi\rangle\big)\big)\big)\leqslant q\big(\varphi\big(\langle\xi,\xi\rangle\big)\big)\\ &= q\big(\big\langle\Phi(\xi),\Phi(\xi)\big\rangle\big)=\bar{q}_{E\otimes_\varphi B}\big(\Phi(\xi)\big)^2\end{aligned}$$

for all $\xi \in E$. Using this fact it is not difficult to check that Φ has closed ranges. On the other hand, for $\xi \in E$ and $b \in B$,

$$\xi \otimes_\varphi b = \lim_i \xi \otimes_\varphi be_i = \lim_i \xi\varphi^{-1}(b) \otimes_\varphi e_i = \Phi\big(\xi\varphi^{-1}(b)\big).$$

Therefore, Φ is surjective and the proposition is proved. □

Corollary 5.7. *Let A, B, E, φ and Φ be as in the above proposition. If $\{h_n\}_n$ is a standard frame of multipliers for $E \otimes_\varphi B$ with the frame operator S, then $\{\Phi^{-1} \circ h_n \circ \varphi\}_n$ is a standard frame of multipliers for E with the frame operator $\Phi^{-1} \circ S \circ \Phi$.*

Corollary 5.8. *Let A, B, E, φ and Φ be as in the above proposition. If $\{\xi_n \otimes_\varphi b_m\}_{(n,m)}$ is a standard frame for $E \otimes_\varphi B$ with the frame operator S, then $\{\xi_n\varphi^{-1}(b_m)\}_{(n,m)}$ is a standard frame of multipliers for E with the frame operator $\Phi^{-1} \circ S \circ \Phi$.*

References

[1] Lj. Arambašić, On frames for countably generated Hilbert C^*-modules, Proc. Amer. Math. Soc. 135 (2) (2007) 469–478.
[2] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952).
[3] M. Fragoulopoulou, Topological Algebras with Involution, North-Holland, Amsterdam, 2005.
[4] M. Frank, D.R. Larson, A module frame concept for Hilbert C^*-modules, in: Functional and Harmonic Analysis of Wavelets, San Antonio, TX, Jan. 1999, in: Contemp. Math., vol. 247, Amer. Math. Soc., Providence, RI, 2000, pp. 207–233.
[5] M. Frank, D.R. Larson, Frames in Hilbert C^*-modules and C^*-algebras, J. Operator Theory 48 (2) (2002) 273–314.
[6] M. Joiţa, Tensor products of Hilbert modules over locally C^*-algebras, Czechoslovak Math. J. 54 (129) (2004) 727–737.
[7] M. Joiţa, Hilbert Modules over Locally C^*-Algebras, University of Bucharest Press, ISBN 973737128-3, 2006, 150 pp.
[8] M. Joiţa, On frames in Hilbert modules over pro-C^*-modules, Topology Appl. 156 (2008) 83–92.
[9] M. Joiţa, Countably generated Hilbert modules and stable isomorphisms of locally C^*-algebras, in: R.G. Douglas, J. Esterle, D. Gaşpar, D. Timotin, F.-H. Vasilescu (Eds.), Hot Topics in Operator Theory, in: Proceedings of the 21st International Conference on Operator Theory, Timisoara, June 29–July 4, 2006, in: Theta Ser. Adv. Math., vol. 9, Theta, Bucharest, 2008, pp. 89–99.
[10] A. Khosravi, B. Khosravi, Frames and bases in tensor products of Hilbert spaces and Hilbert C^*-modules, Proc. Indian Acad. Sci. Math. Sci. 117 (1) (2007) 1–12.
[11] E.C. Lance, Hilbert C^*-Modules. A Toolkit for Operator Algebraists, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge University Press, Cambridge, 1995.
[12] N.C. Phillips, Inverse limits of C^*-algebras, J. Operator Theory 19 (1988) 159–195.
[13] I. Raeburn, S.J. Thompson, Countably generated Hilbert modules, the Kasparov stabilisation theorem, and frames with Hilbert modules, Proc. Amer. Math. Soc. 131 (5) (2003) 1557–1564.